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Multidimensional Toda type systems

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Abstract

On the base of Lie algebraic and differential geometry methods, a wide class of multidimensional nonlinear systems is obtained, and the integration scheme for such equations is proposed.

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1 Introduction

In the present paper we give a Lie algebraic and differential geometry derivation of a wide class of multidimensional nonlinear systems. The systems under consideration are generated by the zero curvature condition for a connection on a trivial principal fiber bundle $M \times G \to M$, constrained by the relevant grading condition. Here M is either the real manifold $\mathbb{R}^{2d}$, or the complex manifold $\mathbb{C}^d$, G is a complex Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$–gradation. We call the arising systems of partial differential equations the multidimensional Toda type systems. From the physical point of view, they describe Toda type fields coupled to matter fields, all of them living on $2d$–dimensional space. Analogously to the two dimensional situation, with an appropriate Inönü–Wigner contraction procedure, one can exclude for our systems the back reaction of the matter fields on the Toda fields.

For the two dimensional case and the finite dimensional Lie algebra $\mathfrak{g}$, connections taking values in the local part of $\mathfrak{g}$ lead to abelian and nonabelian conformal Toda systems and their affine deformations for the affine $\mathfrak{g}$, see [15] and references therein, and also [16, 17] for differential and algebraic geometry background of such systems. For the connection with values in higher grading subspaces of $\mathfrak{g}$ one deals with systems discussed in [12, 9].

In higher dimensions our systems, under some additional specialisations, contain as particular cases the Cecotti–Vafa type equations [5], see also [8]; and those of Gervais–Matsuo [11] which represent some reduction of a generalised WZNW model. Note that some of the arising systems are related to classical problems of differential geometry, coinciding with the well known completely integrable Bourlet type equations [7, 3, 2] and those sometimes called multidimensional generalisation of the sine–Gordon and wave equations, see, for example, [2, 20, 18, 1].

In the paper by the integrability of a system of partial differential equations we mean the existence of a constructive procedure to obtain its general solution. Following the lines of [15, 16, 12, 17], we formulate the integration scheme for the multidimensional Toda type systems. In accordance with this scheme, the multidimensional Toda type and matter type fields are reconstructed from some mappings which we call integration data. In the case when M is $\mathbb{C}^d$, the integration data are divided into holomorphic and antiholomorphic ones; when M is $\mathbb{R}^{2d}$ they depend on one or another half of the independent variables. Moreover, in a multidimensional case the integration data are submitted to the relevant integrability conditions which are absent in the two dimensional situation. These conditions split into two systems of multidimensional nonlinear equations for integration data. If the integrability conditions are integrable systems, then the corresponding multidimensional Toda type system is also integrable. We show that in this case any solution of our systems can be obtained using the proposed integration scheme. It is also investigated when different sets of integration data give the same solution.

Note that the results obtained in the present paper can be extended in a natural way to the case of supergroups.

2 Derivation of equations

In this section we give a derivation of some class of multidimensional nonlinear equations. Our strategy here is a direct generalisation of the method which was used to obtain the Toda type equations in two dimensional case [15, 16, 12, 17]. It consists of the following main steps. We consider a general flat connection on a trivial principal fiber bundle and suppose that the corresponding Lie algebra is endowed with a $\mathbb{Z}$–gradation. Then we impose on the connection some grading conditions and prove that an appropriate gauge transformation allows to bring it to the form parametrised by a set of Toda type and matter type fields. The zero curvature condition for such a connection is equivalent to a set of equations for the fields, which are called the multidimensional Toda type equations. In principle, the form of the equations in question can be postulated. However, the derivation given below suggests also the method of solving these equations, which is explicitly formulated and discussed in section 3.

2.1 Flat connections and gauge transformations

Let M be the manifold $\mathbb{R}^{2d}$ or the manifold $\mathbb{C}^d$. Denote by z^{-i}, z^{+i}, $i = 1, \ldots, d$, the standard coordinates on M. In the case when M is $\mathbb{C}^d$ we suppose that $z^{+i} = \overline{z^{-i}}$. Let G be a complex connected matrix Lie group. The generalisation of the construction given below to the case of a general finite dimensional Lie group is straightforward, see in this connection [16, 17] where such a generalisation was done for the case of two dimensional space M. The general discussion given below can be also well applied to infinite dimensional Lie groups. Consider the trivial principal fiber G–bundle $M \times G \to M$. Denote by $\mathfrak{g}$ the Lie algebra of G. It is well known that there is a bijective correspondence between connection forms on $M \times G \to G$ and $\mathfrak{g}$–valued 1–forms on M. Having in mind this correspondence, we call a $\mathfrak{g}$–valued 1–form on M a connection form, or simply a connection. The curvature 2–form of a connection ω is determined by the 2–form Ω on M, related to ω by the formula

$$\Omega = d\omega + \omega \wedge \omega,$$

and the connection ω is flat if and only if

$$d\omega + \omega \wedge \omega = 0. \tag{2.1}$$

Relation (2.1) is called the *zero curvature condition.*

Let φ be a mapping from M to G. The connection ω of the form

$$\omega = \varphi^{-1} d\varphi$$

satisfies the zero curvature condition. In this case one says that the connection ω is generated by the mapping φ. Since the manifold M is simply connected, any flat connection is generated by some mapping $\varphi : M \to G$.

The gauge transformations of a connection in the case under consideration are described by smooth mappings from M to G. Here for any mapping $\psi : M \to G$, the gauge transformed connection ω^ψ is given by

$$\omega^\psi = \psi^{-1} \omega \psi + \psi^{-1} d\psi. \tag{2.2}$$

Clearly, the zero curvature condition is invariant with respect to the gauge transformations. In other words, if a connection satisfies this condition, then the gauge transformed connection also satisfies this condition. Actually, if a flat connection ω is generated by a mapping φ then the gauge transformed connection ω^ψ is generated by the mapping $\varphi\psi$. It is convenient to call the gauge transformations defined by (2.2), *G–gauge transformations.*

In what follows we deal with a general connection ω satisfying the zero curvature condition. Write for ω the representation

$$\omega = \sum_{i=1}^{d} (\omega_{-i} dz^{-i} + \omega_{+i} dz^{+i}),$$

where $\omega_{\pm i}$ are some mappings from M to $\mathfrak{g}$, called the components of ω. In terms of $\omega_{\pm i}$ the zero curvature condition takes the form

$$\partial_{-i}\omega_{-j} - \partial_{-j}\omega_{-i} + [\omega_{-i}, \omega_{-j}] = 0, \tag{2.3}$$
$$\partial_{+i}\omega_{+j} - \partial_{+j}\omega_{+i} + [\omega_{+i}, \omega_{+j}] = 0, \tag{2.4}$$
$$\partial_{-i}\omega_{+j} - \partial_{+j}\omega_{-i} + [\omega_{-i}, \omega_{+j}] = 0. \tag{2.5}$$

Here and in what follows we use the notation

$$\partial_{-i} = \partial/\partial z^{-i}, \qquad \partial_{+i} = \partial/\partial z^{+i}.$$

Choosing a basis in $\mathfrak{g}$ and treating the components of the expansion of $\omega_{\pm i}$ over this basis as fields, we can consider the zero curvature condition as a nonlinear system of partial differential equations for the fields. Since any flat connection can be gauge transformed to zero, system (2.3)–(2.5) is, in a sense, trivial. From the other hand, we can obtain from (2.3)–(2.5) nontrivial integrable systems by imposing some gauge noninvariant constraints on the connection ω. Consider one of the methods to impose the constraints in question, which is, in fact, a direct generalisation of the group–algebraic approach [15, 16, 12, 17] which was used successfully in two dimensional case ($d = 1$).

2.2 $\mathbb{Z}$–gradations and modified Gauss decomposition

Suppose that the Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}$–graded Lie algebra. This means that $\mathfrak{g}$ is represented as the direct sum

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m, \tag{2.6}$$

where the subspaces $\mathfrak{g}_m$ satisfy the condition

$$[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$$

for all $m, n \in \mathbb{Z}$. It is clear that the subspaces $\mathfrak{g}_0$ and

$$\widetilde{\mathfrak{n}}_- = \bigoplus_{m<0} \mathfrak{g}_m, \qquad \widetilde{\mathfrak{n}}_+ = \bigoplus_{m>0} \mathfrak{g}_m$$

are subalgebras of $\mathfrak{g}$. Denoting the subalgebra $\mathfrak{g}_0$ by $\widetilde{\mathfrak{h}}$, we write the generalised triangle decomposition for $\mathfrak{g}$,

$$\mathfrak{g} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+.$$

Here and in what follows we use tildes to have the notations different from ones usually used for the case of the canonical gradation of a complex semisimple Lie algebra. Note, that this gradation is closely related to the so called principal three–dimensional subalgebra of the Lie algebra under consideration [4, 17].

Denote by $\widetilde{H}$ and by $\widetilde{N}_\pm$ the connected Lie subgroups corresponding to the subalgebras $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_\pm$. Suppose that $\widetilde{H}$ and $\widetilde{N}_\pm$ are closed subgroups of G and, moreover,

$$\widetilde{H} \cap \widetilde{N}_\pm = \{e\}, \qquad \widetilde{N}_- \cap \widetilde{N}_+ = \{e\}, \tag{2.7}$$

$$\widetilde{N}_- \cap \widetilde{H}\widetilde{N}_+ = \{e\}, \qquad \widetilde{N}_-\widetilde{H} \cap \widetilde{N}_+ = \{e\}. \tag{2.8}$$

where e is the unit element of G. This is true, in particular, for the reductive Lie groups, see, for example, [13]. The set $\widetilde{N}_-\widetilde{H}\widetilde{N}_+$ is an open subset of G. Suppose that

$$G = \overline{\widetilde{N}_-\widetilde{H}\widetilde{N}_+}.$$

This is again true, in particular, for the reductive Lie groups. Thus, for an element a belonging to the dense subset of G, one has the following, convenient for our aims, decomposition:

$$a = n_- h n_+^{-1}, \tag{2.9}$$

where $n_\pm \in \widetilde{N}_\pm$ and $h \in \widetilde{H}$. Decomposition (2.9) is called the *Gauss decomposition*. Due to (2.7) and (2.8), this decomposition is unique. Actually, (2.9) is one of the possible forms of the Gauss decomposition. Taking the elements belonging to the subgroups $\widetilde{N}_\pm$ and $\widetilde{H}$ in different orders we get different types of the Gauss decompositions valid in the corresponding dense subsets of G. In particular, below, besides of decomposition (2.9), we will often use the Gauss decompositions of the forms

$$a = m_- n_+ h_+, \qquad a = m_+ n_- h_-, \tag{2.10}$$

where $m_\pm \in \widetilde{N}_\pm$, $n_\pm \in \widetilde{N}_\pm$ and $h_\pm \in \widetilde{H}$. The main disadvantage of any form of the Gauss decomposition is that not any element of G possesses such a decomposition. To overcome this difficulty, let us consider so called modified Gauss decompositions. They are based on the following almost trivial remark. If an element $a \in G$ does not admit the Gauss decomposition of some form, then, subjecting a to some left shift in G, we can easily get an element admitting that decomposition. So, in particular, we can say that any element of G can be represented in forms (2.10) where $m_\pm \in a_\pm \widetilde{N}_\pm$ for some elements $a_\pm \in G$, $n_\pm \in \widetilde{N}_\pm$ and $h_\pm \in \widetilde{H}$. If the elements $a_\pm$ are fixed, then decompositions (2.10) are unique. We call the Gauss decompositions obtained in such a way, the *modified Gauss decompositions* [16, 17].

Let $\varphi : M \to G$ be an arbitrary mapping and p be an arbitrary point of M. Suppose that $a_\pm$ are such elements of G that the element $\varphi(p)$ admits the modified Gauss decompositions (2.10). It can be easily shown that for any point p' belonging to some neighborhood of p, the element $\varphi(p')$ admits the modified Gauss decompositions (2.10) for the same choice of

the elements $a_\pm$ [16, 17]. In other words, any mapping $\varphi : M \to G$ has the following local decompositions

$$\varphi = \mu_+ \nu_- \eta_-, \qquad \varphi = \mu_- \nu_+ \eta_+, \tag{2.11}$$

where the mappings $\mu_\pm$ take values in $a_\pm \widetilde{N}_\pm$ for some elements $a_\pm \in G$, the mappings $\nu_\pm$ take values in $\widetilde{N}_\pm$, and the mappings $\eta_\pm$ take values in $\widetilde{H}$. It is also clear that the mappings $\mu_+^{-1} \partial_{\pm i} \mu_+$ take values in $\widetilde{\mathfrak{n}}_+$, while the mappings $\mu_-^{-1} \partial_{\pm i} \mu_-$ take values in $\widetilde{\mathfrak{n}}_-$.

2.3 Grading conditions

The first condition we impose on the connection ω is that the components ω_{-i} take values in $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$, and the components ω_{+i} take values in $\widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+$. We call this condition the *general grading condition.*

Let a mapping $\varphi : M \to G$ generates the connection ω; in other words, $\omega = \varphi^{-1} d\varphi$. Using respectively the first and the second equalities from (2.11), we can write the following representations for the connection components ω_{-i} and ω_{+i}:

$$\omega_{-i} = \eta_-^{-1} \nu_-^{-1} (\mu_+^{-1} \partial_{-i} \mu_+) \nu_- \eta_- + \eta_-^{-1} (\nu_-^{-1} \partial_{-i} \nu_-) \eta_- + \eta_-^{-1} \partial_{-i} \eta_-, \tag{2.12}$$

$$\omega_{+i} = \eta_+^{-1} \nu_+^{-1} (\mu_-^{-1} \partial_{+i} \mu_-) \nu_+ \eta_+ + \eta_+^{-1} (\nu_+^{-1} \partial_{+i} \nu_+) \eta_+ + \eta_+^{-1} \partial_{+i} \eta_+. \tag{2.13}$$

From these relations it follows that the connection ω satisfies the general grading condition if and only if

$$\partial_{\pm i} \mu_\mp = 0. \tag{2.14}$$

When $M = \mathbb{R}^{2d}$ these equalities mean that μ_- depends only on coordinates z^{-i}, and μ_+ depends only on coordinates z^{+i}. When $M = \mathbb{C}^d$ they mean that μ_- is a holomorphic mapping, and μ_+ is an antiholomorphic one. For a discussion of the differential geometry meaning of the general grading condition, which is here actually the same as for two dimensional case, we refer the reader to [16, 17].

Perform now a further specification of the grading condition. Define the subspaces $\widetilde{\mathfrak{m}}_{\pm i}$ of $\widetilde{\mathfrak{n}}_\pm$ by

$$\widetilde{\mathfrak{m}}_{-i} = \bigoplus_{-l_{-i} \le m \le -1} \mathfrak{g}_m, \qquad \widetilde{\mathfrak{m}}_{+i} = \bigoplus_{1 \le m \le l_{+i}} \mathfrak{g}_m,$$

where $l_{\pm i}$ are some positive integers. Let us require that the connection components ω_{-i} take values in the subspace $\widetilde{\mathfrak{m}}_{-i} \oplus \widetilde{\mathfrak{h}}$, and the components ω_{+i} take values in $\widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{m}}_{+i}$. We call such a requirement the *specified grading condition.* Using the modified Gauss decompositions (2.11), one gets

$$\omega_{-i} = \eta_+^{-1} \nu_+^{-1} (\mu_-^{-1} \partial_{-i} \mu_-) \nu_+ \eta_+ + \eta_+^{-1} (\nu_+^{-1} \partial_{-i} \nu_+) \eta_+ + \eta_+^{-1} \partial_{-i} \eta_+, \tag{2.15}$$

$$\omega_{+i} = \eta_-^{-1} \nu_-^{-1} (\mu_+^{-1} \partial_{+i} \mu_+) \nu_- \eta_- + \eta_-^{-1} (\nu_-^{-1} \partial_{+i} \nu_-) \eta_- + \eta_-^{-1} \partial_{+i} \eta_-. \tag{2.16}$$

Here the second equality from (2.11) was used for ω_{-i} and the first one for ω_{+i}. From relations (2.15) and (2.16) we conclude that the connection ω satisfies the specified grading condition if and only if the mappings $\mu_-^{-1} \partial_{-i} \mu_-$ take values in $\widetilde{\mathfrak{m}}_{-i}$, and the mappings $\mu_+^{-1} \partial_{+i} \mu_+$ take values in $\widetilde{\mathfrak{m}}_{+i}$.

It is clear that the general grading condition and the specified grading condition are not invariant under the action of an arbitrary G–gauge transformation, but they are invariant

under the action of gauge transformations (2.2) with the mapping ψ taking values in the subgroup $\widetilde{H}$. In other words, the system arising from the zero curvature condition for the connection satisfying the specified grading condition still possesses some gauge symmetry. Below we call a gauge transformation (2.2) with the mapping ψ taking values in $\widetilde{H}$ an $\widetilde{H}$–*gauge transformations*. Let us impose now one more restriction on the connection and use the $\widetilde{H}$–gauge symmetry to bring it to the form generating equations free of the $\widetilde{H}$–gauge invariance.

2.4 Final form of connection

Taking into account the specified grading condition, we write the following representation for the components of the connection ω:

$$\omega_{-i} = \sum_{m=0}^{-l_{-i}} \omega_{-i,m}, \qquad \omega_{+i} = \sum_{m=0}^{l_{+i}} \omega_{+i,m},$$

where the mappings $\omega_{\pm i,m}$ take values in $\mathfrak{g}_{\pm m}$. There is a similar decomposition for the mappings $\mu_\pm^{-1}\partial_{\pm i}\mu_\pm$:

$$\mu_-^{-1}\partial_{-i}\mu_- = \sum_{m=-1}^{-l_{-i}} \lambda_{-i,m}, \qquad \mu_+^{-1}\partial_{+i}\mu_+ = \sum_{m=1}^{l_{+i}} \lambda_{+i,m}.$$

From (2.15) and (2.16) it follows that

$$\omega_{\pm i,\pm l_{\pm i}} = \eta_\mp^{-1}\lambda_{\pm i,\pm l_{\pm i}}\eta_\mp. \tag{2.17}$$

The last restriction we impose on the connection ω is formulated as follows. Let $c_{\pm i}$ be some fixed elements of the subspaces $\mathfrak{g}_{\pm l_{\pm i}}$ satisfying the relations

$$[c_{-i}, c_{-j}] = 0, \qquad [c_{+i}, c_{+j}] = 0. \tag{2.18}$$

Require that the mappings $\omega_{\pm i,\pm l_{\pm i}}$ have the form

$$\omega_{\pm i,\pm l_{\pm i}} = \eta_\mp^{-1}\gamma_\pm c_{\pm i}\gamma_\pm^{-1}\eta_\mp \tag{2.19}$$

for some mappings $\gamma_\pm : M \to \widetilde{H}$. A connection which satisfies the grading condition and relation (2.19) is called an *admissible connection*. Similarly, a mapping from M to G generating an admissible connection is called *an admissible mapping*. Taking into account (2.17), we conclude that

$$\lambda_{\pm i,\pm l_{\pm i}} = \gamma_\pm c_{\pm i}\gamma_\pm^{-1}. \tag{2.20}$$

Denote by $\widetilde{H}_-$ and $\widetilde{H}_+$ the isotropy subgroups of the sets formed by the elements c_{-i} and c_{+i}, respectively. It is clear that the mappings $\gamma_\pm$ are defined up to multiplication from the right side by mappings taking values in $\widetilde{H}_\pm$. In any case, at least locally, we can choose the mappings $\gamma_\pm$ in such a way that

$$\partial_{\mp i}\gamma_\pm = 0. \tag{2.21}$$

In what follows we use such a choice for the mappings $\gamma_\pm$.

Let us show now that there exists a local $\widetilde{H}$–gauge transformation that brings an admissible connection to the connection ω with the components of the form

$$\omega_{-i} = \gamma^{-1}\partial_{-i}\gamma + \sum_{m=-1}^{-l_{-i}+1} \upsilon_{-i,m} + c_{-i}, \tag{2.22}$$

$$\omega_{+i} = \gamma^{-1}\left(\sum_{m=1}^{l_{+i}-1} \upsilon_{+i,m} + c_{+i}\right)\gamma, \tag{2.23}$$

where γ is some mapping from M to $\widetilde{H}$, and $\upsilon_{\pm i,m}$ are mappings taking values in $\mathfrak{g}_{\pm m}$.

To prove the above statement, note first that taking into account (2.14), we get from (2.12) and (2.13) the following relations

$$\omega_{-i} = \eta_-^{-1}(\nu_-^{-1}\partial_{-i}\nu_-)\eta_- + \eta_-^{-1}\partial_{-i}\eta_-, \tag{2.24}$$

$$\omega_{+i} = \eta_+^{-1}(\nu_+^{-1}\partial_{+i}\nu_+)\eta_+ + \eta_+^{-1}\partial_{+i}\eta_+. \tag{2.25}$$

Comparing (2.24) and (2.15), we come to the relation

$$\nu_-^{-1}\partial_{-i}\nu_- = \left[\eta\nu_+^{-1}(\mu_-^{-1}\partial_{-i}\mu_-)\nu_+\eta^{-1}\right]_{\widetilde{\mathfrak{n}}_-},$$

where

$$\eta = \eta_-\eta_+^{-1}. \tag{2.26}$$

Hence, the mappings $\nu_-^{-1}\partial_{-i}\nu_-$ take values in subspaces $\mathfrak{m}_{-i}$ and we can represent them in the form

$$\nu_-^{-1}\partial_{-i}\nu_- = \eta\gamma_-\left(\sum_{m=-1}^{-l_{-i}} \upsilon_{-i,m}\right)\gamma_-^{-1}\eta^{-1},$$

with the mappings $\upsilon_{-i,m}$ taking values in $\mathfrak{g}_{-m}$. Substituting this representation into (2.24), we obtain

$$\omega_{-i} = \eta_+^{-1}\gamma_-\left(\sum_{m=-1}^{-l_{-i}} \upsilon_{-i,m}\right)\gamma_-^{-1}\eta_+ + \eta_-^{-1}\partial_{-i}\eta_-.$$

From (2.17) and (2.20) it follows that $\upsilon_{-i,-l_{-i}} = c_{-i}$. Therefore,

$$\omega_{-i} = \eta_+^{-1}\gamma_-\left(c_{-i} + \sum_{m=-1}^{-l_{-i}+1} \upsilon_{-i,m}\right)\gamma_-^{-1}\eta_+ + \eta_-^{-1}\partial_{-i}\eta_-. \tag{2.27}$$

Similarly, using (2.25) and (2.16), we conclude that

$$\nu_+^{-1}\partial_{+i}\nu_+ = \left[\eta^{-1}\nu_-^{-1}(\mu_+^{-1}\partial_{+i}\mu_+)\nu_-\eta\right]_{\widetilde{\mathfrak{n}}_+}.$$

Therefore we can write for $\nu_+^{-1}\partial_{+i}\nu_+$ the representation

$$\nu_+^{-1}\partial_{+i}\nu_+ = \eta^{-1}\gamma_+\left(\sum_{m=1}^{l_{+i}} \upsilon_{+i,m}\right)\gamma_+^{-1}\eta,$$

where the mappings $\upsilon_{+i,m}$ take values in $\mathfrak{g}_m$. Taking into account (2.25), we get

$$\omega_{+i} = \eta_-^{-1}\gamma_+ \left(\sum_{m=1}^{l_{+i}} \upsilon_{+i,m} \right) \gamma_+^{-1}\eta_- + \eta_+^{-1}\partial_{+i}\eta_+.$$

Using again (2.17) and (2.20), we obtain $\upsilon_{+i,l_{+i}} = c_{+i}$. Therefore, the following relation is valid:

$$\omega_{+i} = \eta_-^{-1}\gamma_+ \left(\sum_{m=1}^{l_{+i}-1} \upsilon_{+i,m} + c_{+i} \right) \gamma_+^{-1}\eta_- + \eta_+^{-1}\partial_{+i}\eta_+. \qquad (2.28)$$

Taking into account (2.27) and (2.28) and performing the gauge transformation defined by the mapping $\eta_+^{-1}\gamma_-$, we arrive at the connection with the components of the form given by (2.22) and (2.23) with

$$\gamma = \gamma_+^{-1}\eta\gamma_-. \qquad (2.29)$$

Note that the connection with components (2.22), (2.23) is generated by the mapping

$$\varphi = \mu_+\nu_-\eta\gamma_- = \mu_-\nu_+\gamma_-. \qquad (2.30)$$

2.5 Multidimensional Toda type equations

The equations for the mappings γ and $\upsilon_{\pm i,m}$, which result from the zero curvature condition (2.3)–(2.5) with the connection components of form (2.22), (2.23), will be called *multidimensional Toda type equations*, or *multidimensional Toda type systems*. It is natural to call the functions parametrising the mappings γ and $\upsilon_{\pm i,m}$, *Toda type* and *matter type fields*, respectively.

The multidimensional Toda type equations are invariant with respect to the remarkable symmetry transformations

$$\gamma' = \xi_+^{-1}\gamma\xi_-, \qquad \upsilon'_{\pm i} = \xi_\pm^{-1}\upsilon_{\pm i}\xi_\pm, \qquad (2.31)$$

where $\xi_\pm$ are arbitrary mappings taking values in the isotropy subgroups $\widetilde{H}_\pm$ of the sets formed by the elements c_{-i} and c_{+i}, and satisfying the relations

$$\partial_\mp\xi_\pm = 0. \qquad (2.32)$$

Indeed, it can be easily verified that the connection components of form (2.22), (2.23) constructed with the mappings γ, $\upsilon_{\pm i}$ and γ', $\upsilon'_{\pm i}$ are connected by the $\widetilde{H}$–gauge transformation generated by the mapping ξ_-. Therefore, if the mappings γ, $\upsilon_{\pm i}$ satisfy the multidimensional Toda type equations, then the mappings γ', $\upsilon'_{\pm i}$ given by (2.31) satisfy the same equations. Note that, because the mappings $\xi_\pm$ are subjected to (2.32), transformations (2.31) are not *gauge* symmetry transformations of the multidimensional Toda type equations.

Let us make one more useful remark. Let $h_\pm$ be some fixed elements of $\widetilde{H}$, and mappings γ, $\upsilon_{\pm i}$ satisfy the multidimensional Toda type equations generated by the connection with the components of form (2.22), (2.23). It is not difficult to get convinced that the mappings

$$\gamma' = h_+^{-1}\gamma h_-, \qquad \upsilon'_{\pm i} = h_\pm^{-1}\upsilon_{\pm i}h_\pm$$

satisfy the multidimensional Toda type equations where instead of $c_{\pm i}$ one uses the elements

$$c'_{\pm i} = h_{\pm}^{-1} c_{\pm i} h_{\pm}.$$

In such a sense, the multidimensional Toda type equations determined by the elements $c_{\pm i}$ and $c'_{\pm i}$ which are connected by the above relation, are equivalent.

Let us write the general form of the multidimensional Toda type equations for $l_{-i} = l_{+i} = 1$ and $l_{-i} = l_{+i} = 2$. The cases with other choices of $l_{\pm i}$ can be treated similarly.

Consider first the case $l_{-i} = l_{+i} = 1$. Here the connection components have the form

$$\omega_{-i} = \gamma^{-1}\partial_{-i}\gamma + c_{-i}, \qquad \omega_{+i} = \gamma^{-1}c_{+i}\gamma.$$

Equations (2.3) are equivalent here to the following ones:

$$[c_{-i}, \gamma^{-1}\partial_{-j}\gamma] + [\gamma^{-1}\partial_{-i}\gamma, c_{-j}] = 0, \tag{2.33}$$
$$\partial_{-i}(\gamma^{-1}\partial_{-j}\gamma) - \partial_{-j}(\gamma^{-1}\partial_{-i}\gamma) + [\gamma^{-1}\partial_{-i}\gamma, \gamma^{-1}\partial_{-j}\gamma] = 0. \tag{2.34}$$

Equations (2.34) are satisfied by any mapping γ; equations (2.33) can be identically rewritten as

$$\partial_{-i}(\gamma c_{-j}\gamma^{-1}) = \partial_{-j}(\gamma c_{-i}\gamma^{-1}). \tag{2.35}$$

Analogously, equations (2.4) read

$$\partial_{+i}(\gamma^{-1}c_{+j}\gamma) = \partial_{+j}(\gamma^{-1}c_{+i}\gamma). \tag{2.36}$$

Finally, we easily get convinced that equations (2.5) can be written as

$$\partial_{+j}(\gamma^{-1}\partial_{-i}\gamma) = [c_{-i}, \gamma^{-1}c_{+j}\gamma]. \tag{2.37}$$

Thus, the zero curvature condition in the case under consideration is equivalent to equations (2.35)–(2.37). In the two dimensional case equations (2.35) and (2.36) are absent, and equations (2.37) take the form

$$\partial_{+}(\gamma^{-1}\partial_{-}\gamma) = [c_{-}, \gamma^{-1}c_{+}\gamma].$$

If the Lie group G is semisimple, then using the canonical gradation of the corresponding Lie algebra $\mathfrak{g}$, we get the well known abelian Toda equations; noncanonical gradations lead to various nonabelian Toda systems.

Proceed now to the case $l_{-i} = l_{+i} = 2$. Here the connection components are

$$\omega_{-i} = \gamma^{-1}\partial_{-i}\gamma + \upsilon_{-i} + c_{-i}, \qquad \omega_{+i} = \gamma^{-1}(\upsilon_{+i} + c_{+i})\gamma,$$

where we have denoted $\upsilon_{\pm i,\pm 1}$ simply by $\upsilon_{\pm i}$. Equations (2.3) take the form

$$[c_{-i}, \upsilon_{-j}] = [c_{-j}, \upsilon_{-i}], \tag{2.38}$$
$$\partial_{-i}(\gamma c_{-j}\gamma^{-1}) - \partial_{-j}(\gamma c_{-i}\gamma^{-1}) = [\gamma\upsilon_{-j}\gamma^{-1}, \gamma\upsilon_{-i}\gamma^{-1}], \tag{2.39}$$
$$\partial_{-i}(\gamma\upsilon_{-j}\gamma^{-1}) = \partial_{-j}(\gamma\upsilon_{-i}\gamma^{-1}). \tag{2.40}$$

The similar system of equations follows from (2.4),

$$[c_{+i}, \upsilon_{+j}] = [c_{+j}, \upsilon_{+i}], \tag{2.41}$$

$$\partial_{+i}(\gamma^{-1} c_{+j} \gamma) - \partial_{+j}(\gamma^{-1} c_{+i} \gamma) = [\gamma^{-1} \upsilon_{+j} \gamma, \gamma^{-1} \upsilon_{+i} \gamma], \tag{2.42}$$

$$\partial_{+i}(\gamma^{-1} \upsilon_{+j} \gamma) = \partial_{+j}(\gamma^{-1} \upsilon_{+i} \gamma). \tag{2.43}$$

After some calculations we get from (2.5) the equations

$$\partial_{-i} \upsilon_{+j} = [c_{+j}, \gamma \upsilon_{-i} \gamma^{-1}], \tag{2.44}$$

$$\partial_{+j} \upsilon_{-i} = [c_{-i}, \gamma^{-1} \upsilon_{+j} \gamma], \tag{2.45}$$

$$\partial_{+j}(\gamma^{-1} \partial_{-i} \gamma) = [c_{-i}, \gamma^{-1} c_{+j} \gamma] + [\upsilon_{-i}, \gamma^{-1} \upsilon_{+j} \gamma]. \tag{2.46}$$

Thus, in the case $l_{-i} = l_{+i} = 2$ the zero curvature condition is equivalent to the system of equations (2.38)–(2.46). In the two dimensional case we come to the equations

$$\partial_- \upsilon_+ = [c_+, \gamma \upsilon_- \gamma^{-1}], \qquad \partial_+ \upsilon_- = [c_-, \gamma^{-1} \upsilon_+ \gamma],$$
$$\partial_+(\gamma^{-1} \partial_- \gamma) = [c_-, \gamma^{-1} c_+ \gamma] + [\upsilon_-, \gamma^{-1} \upsilon_+ \gamma],$$

which represent the simplest case of higher grading Toda systems [12].

3 Construction of general solution

From the consideration presented above it follows that any admissible mapping generates local solutions of the corresponding multidimensional Toda type equations. Thus, if we were to be able to construct admissible mappings, we could construct solutions of the multidimensional Toda type equations. It is worth to note here that the solutions in questions are determined by the mappings $\mu_\pm$, $\nu_\pm$ entering Gauss decompositions (2.11), and from the mapping η which is defined via (2.26) by the mappings $\eta_\pm$ entering the same decomposition. So, the problem is to find the mappings $\mu_\pm$, $\nu_\pm$ and η arising from admissible mappings by means of Gauss decompositions (2.11) and relation (2.26). It appears that this problem has a remarkably simple solution.

Recall that a mapping $\varphi : M \to G$ is admissible if and only if the mappings $\mu_\pm$ entering Gauss decompositions (2.11) satisfy conditions (2.14), and the mappings $\mu_\pm^{-1} \partial_{\pm i} \mu_\pm$ have the form

$$\mu_-^{-1} \partial_{-i} \mu_- = \gamma_- c_{-i} \gamma_-^{-1} + \sum_{m=-1}^{-l_{-i}+1} \lambda_{-i,m}, \tag{3.1}$$

$$\mu_+^{-1} \partial_{+i} \mu_+ = \sum_{m=1}^{l_{+i}-1} \lambda_{+i,m} + \gamma_+ c_{+i} \gamma_+^{-1}. \tag{3.2}$$

Here $\gamma_\pm$ are some mappings taking values in $\widetilde{H}$ and satisfying conditions (2.21); the mappings $\lambda_{\pm i,m}$ take values in $\mathfrak{g}_{\pm m}$; and $c_{\pm i}$ are the fixed elements of the subspaces $\mathfrak{g}_{\pm l_\pm}$, which satisfy relations (2.18).

From the other hand, the mappings $\mu_\pm$ uniquely determine the mappings $\nu_\pm$ and η. Indeed, from (2.11) one gets

$$\mu_+^{-1}\mu_- = \nu_-\eta\nu_+^{-1}. \tag{3.3}$$

Relation (3.3) can be considered as the Gauss decomposition of the mapping $\mu_+^{-1}\mu_-$ induced by the Gauss decomposition (2.9). Hence, the mappings $\mu_\pm$ uniquely determine the mappings $\nu_\pm$ and η.

Taking all these remarks into account we propose the following procedure for obtaining solutions to the multidimensional Toda type equations.

3.1 Integration scheme

Let $\gamma_\pm$ be some mappings taking values in $\widetilde{H}$, and $\lambda_{\pm i,m}$ be some mappings taking values in $\mathfrak{g}_{\pm m}$. Here it is supposed that

$$\partial_{\mp i}\gamma_\pm = 0, \qquad \partial_{\mp i}\lambda_{\pm j,m} = 0. \tag{3.4}$$

Consider (3.1) and (3.2) as a system of partial differential equations for the mappings $\mu_\pm$ and try to solve it. Since we are going to use the mappings $\mu_\pm$ for construction of admissible mappings, we have to deal only with solutions of equations (3.1) and (3.2) which satisfy relations (2.14). The latter are equivalent to the following ones:

$$\mu_-^{-1}\partial_{+i}\mu_- = 0, \tag{3.5}$$
$$\mu_+^{-1}\partial_{-i}\mu_+ = 0. \tag{3.6}$$

So, we have to solve the system consisting of equations (3.1), (3.2) and (3.5), (3.6). Certainly, it is possible to solve this system if and only if the corresponding integrability conditions are satisfied. The right hand sides of equations (3.1), (3.5) and (3.2), (3.6) can be interpreted as components of flat connections on the trivial principal fiber bundle $M \times G \to M$. Therefore, the integrability conditions of equations (3.1), (3.5) and (3.2), (3.6) look as the zero curvature condition for these connections. In particular, for the case $l_{-i} = l_{+i} = 2$ the integrability conditions are

$$\begin{gathered}\partial_{\pm i}\lambda_{\pm j} = \partial_{\pm j}\lambda_{\pm i}, \\ \partial_{\pm i}(\gamma_\pm c_{\pm j}\gamma_\pm^{-1}) - \partial_{\pm j}(\gamma_\pm c_{\pm i}\gamma_\pm^{-1}) = [\lambda_{\pm j}, \lambda_{\pm i}], \\ [\lambda_{\pm i}, \gamma_\pm c_{\pm j}\gamma_\pm^{-1}] = [\lambda_{\pm j}, \gamma_\pm c_{\pm i}\gamma_\pm^{-1}],\end{gathered}$$

where we have denoted $\lambda_{\pm i,1}$ simply by $\lambda_{\pm i}$.

In general, the integrability conditions can be considered as two systems of partial nonlinear differential equations for the mappings γ_-, $\lambda_{-i,m}$ and γ_+, $\lambda_{+i,m}$, respectively. The multidimensional Toda type equations are integrable if and only if these systems are integrable. In any case, if we succeed to find a solution of the integrability conditions, we can construct the corresponding solution of the multidimensional Toda type equations. A set of mappings $\gamma_\pm$ and $\lambda_{\pm i,m}$ satisfying (3.4) and the corresponding integrability conditions will be called *integration data*. It is clear that for any set of integration data the solution of equations (3.1), (3.5) and (3.2), (3.6) is fixed by the initial conditions which are constant

elements of the group G. More precisely, let p be some fixed point of M and $a_\pm$ be some fixed elements of G. Then there exists a unique solution of equations (3.1), (3.5) and (3.2), (3.6) satisfying the conditions

$$\mu_\pm(p) = a_\pm. \tag{3.7}$$

It is not difficult to show that the mappings $\mu_\pm$ satisfying the equations under consideration and initial conditions (3.7) take values in $a_\pm \widetilde{N}_\pm$. Note that in the two dimensional case the integrability conditions become trivial.

The next natural step is to use Gauss decomposition (3.3) to obtain the mappings $\nu_\pm$ and η. In general, solving equations (3.1), (3.5) and (3.2), (3.6), we get the mappings $\mu_\pm$, for which the mapping $\mu_+^{-1}\mu_-$ may have not the Gauss decomposition of form (3.3) at some points of M. In such a case one comes to solutions of the multidimensional Toda type equations with some irregularities.

Having found the mappings $\mu_\pm$ and η, one uses (2.29) and the relations

$$\sum_{m=-1}^{-l_{-i}} \upsilon_{-i,m} = \gamma_-^{-1}\eta^{-1}(\nu_-^{-1}\partial_{-i}\nu_-)\eta\gamma_-, \tag{3.8}$$

$$\sum_{m=1}^{l_{+i}} \upsilon_{+i,m} = \gamma_+^{-1}\eta(\nu_+^{-1}\partial_{+i}\nu_+)\eta^{-1}\gamma_+ \tag{3.9}$$

to construct the mappings γ and $\upsilon_{\pm i,m}$. Show that these mappings satisfy the multidimensional Toda type equations. To this end consider the mapping

$$\varphi = \mu_+\nu_-\eta\gamma_- = \mu_-\nu_+\gamma_-,$$

whose form is actually suggested by (2.30). The mapping φ is admissible. Moreover, using formulas of section 2, it is not difficult to demonstrate that it generates the connection with components of form (2.22) and (2.23), where the mappings γ and $\upsilon_{\pm i,m}$ are defined by the above construction. Since this connection is certainly flat, the mappings γ and $\upsilon_{\pm i,m}$ satisfy the multidimensional Toda type equations.

3.2 Generality of solution

Prove now that any solution of the multidimensional Toda type equations can be obtained by the integration scheme described above. Let $\gamma : M \to \widetilde{H}$ and $\upsilon_{\pm i,m} : M \to \mathfrak{g}_{\pm m}$ be arbitrary mappings satisfying the multidimensional Toda type equations. We have to show that there exists a set of integration data leading, by the above integration scheme, to the mappings γ and $\upsilon_{\pm i,m}$.

Using γ and $\upsilon_{\pm i,m}$, construct the connection with the components given by (2.22) and (2.23). Since this connection is flat and admissible, there exists an admissible mapping $\varphi : M \to G$ which generates it. Write for φ local Gauss decompositions (2.11). The mappings $\mu_\pm$ entering these decompositions satisfy relations (2.14). Since the mapping φ is admissible, we have expansions (3.1), (3.2). It is convenient to write them in the form

$$\mu_-^{-1}\partial_{-i}\mu_- = \gamma'_- c_{-i}\gamma'^{-1}_- + \sum_{m=-1}^{-l_{-i}+1} \lambda_{-i,m}, \tag{3.10}$$

$$\mu_+^{-1}\partial_{+i}\mu_+ = \sum_{m=1}^{l_{+i}-1} \lambda_{+i,m} + \gamma'_+ c_{+i} \gamma'^{-1}_+, \tag{3.11}$$

where we use primes because, in general, the mappings $\gamma'_\pm$ are not yet the mappings leading to the considered solution of the multidimensional Toda type equations. Choose the mappings $\gamma'_\pm$ in such a way that

$$\partial_{\mp i}\gamma'_\pm = 0.$$

Formulas (2.27) and (2.28) take in our case the form

$$\omega_{-i} = \eta_+^{-1}\gamma'_- \left(c_{-i} + \sum_{m=-1}^{-l_{-i}+1} \upsilon'_{-i,m} \right) \gamma'^{-1}_- \eta_+ + \eta_-^{-1}\partial_{-i}\eta_-, \tag{3.12}$$

$$\omega_{+i} = \eta_-^{-1}\gamma'_+ \left(\sum_{m=1}^{l_{+i}-1} \upsilon'_{+i,m} + c_{+i} \right) \gamma'^{-1}_+ \eta_- + \eta_+^{-1}\partial_{+i}\eta_+, \tag{3.13}$$

where the mappings $\upsilon'_{\pm i,m}$ are defined by the relations

$$\sum_{m=-1}^{-l_{-i}} \upsilon'_{-i,m} = \gamma'^{-1}_- \eta^{-1}(\nu_-^{-1}\partial_{-i}\nu_-)\eta\gamma'_-, \tag{3.14}$$

$$\sum_{m=1}^{l_{+i}} \upsilon'_{+i,m} = \gamma'^{-1}_+ \eta(\nu_+^{-1}\partial_{+i}\nu_+)\eta^{-1}\gamma'_+. \tag{3.15}$$

From (3.13) and (2.23) it follows that the mapping η_+ satisfies the relation

$$\partial_{+i}\eta_+ = 0.$$

Therefore, for the mapping

$$\xi_- = \gamma'^{-1}_- \eta_+$$

one has

$$\partial_{+i}\xi_- = 0.$$

Comparing (3.12) and (2.22) one sees that the mapping ξ_- takes values in $\widetilde{H}_-$. Relation (2.30) suggests to define

$$\gamma_- = \eta_+, \tag{3.16}$$

thereof

$$\gamma_- = \gamma'_-\xi_-.$$

Further, from (3.12) and (2.22) we conclude that

$$\partial_{-i}(\eta_-\gamma^{-1}) = 0,$$

and, hence, for the mapping

$$\xi_+ = \gamma'^{-1}_+ \eta_- \gamma^{-1}$$

one has

$$\partial_{-i}\xi_+ = 0.$$

Comparing (3.13) and (2.23), we see that the mapping ξ_+ takes values in $\widetilde{H}_+$. Denoting

$$\gamma_+ = \eta_-\gamma^{-1}, \tag{3.17}$$

we get

$$\gamma_+ = \gamma'_+\xi_+.$$

Show now that the mappings $\gamma_\pm$ we have just defined, and the mappings $\lambda_{\pm i,m}$ determined by relations (3.10), (3.11), are the sought for mappings leading to the considered solution of the multidimensional Toda type equations.

Indeed, since the mappings $\xi_\pm$ take values in $\widetilde{H}_\pm$, one gets from (3.10) and (3.11) that the mappings $\mu_\pm$ can be considered as solutions of equations (3.1) and (3.2). Further, the mappings $\nu_\pm$ and $\eta = \eta_-\eta_+^{-1}$ can be treated as the mappings obtained from the Gauss decomposition (3.3). Relations (3.16) and (3.17) imply that the mapping γ is given by (2.29). Now, from (3.12), (3.13) and (2.22), (2.23) it follows that

$$\upsilon_{\pm i,m} = \xi_\pm^{-1}\upsilon'_{\pm i,m}\xi_\pm.$$

Taking into account (3.14) and (3.15), we finally see that the mappings $\upsilon_{\pm i,m}$ satisfy relations (3.8) and (3.9). Thus, any solution of the multidimensional Toda type equations can be locally obtained by the above integration scheme.

3.3 Dependence of solution on integration data

It appears that different sets of integration data can give the same solution of the multidimensional Toda type equations. Consider this problem in detail. Let $\gamma_\pm$, $\lambda_{\pm i,m}$ and $\gamma'_\pm$, $\lambda'_{\pm i,m}$ be two sets of mappings satisfying the integrability conditions of the equations determining the corresponding mappings $\mu_\pm$ and $\mu'_\pm$. Suppose that the solutions γ, $\upsilon_{\pm i,m}$ and γ', $\upsilon'_{\pm i,m}$ obtained by the above procedure coincide. In this case the corresponding connections ω and ω' also coincide. As it follows from the discussion given in section 3.1, these connections are generated by the mappings φ and φ' defined as

$$\varphi = \mu_-\nu_+\gamma_- = \mu_+\nu_-\eta\gamma_-, \qquad \varphi' = \mu'_-\nu'_+\gamma'_- = \mu'_+\nu'_-\eta'\gamma'_-. \tag{3.18}$$

Since the connections ω and ω' coincide, we have

$$\varphi' = a\varphi$$

for some element $a \in G$. Hence, from (3.18) it follows that

$$\mu'_-\nu'_+\gamma'_- = a\mu_-\nu_+\gamma_-.$$

This equality can be rewritten as

$$\mu'_- = a\mu_-\chi_+\psi_+, \tag{3.19}$$

where the mappings χ_+ and ψ_+ are defined by

$$\chi_+ = \nu_+ \gamma_- \gamma_-'^{-1} \nu_+'^{-1} \gamma_-' \gamma_-^{-1}, \qquad \psi_+ = \gamma_- \gamma_-'^{-1}.$$

Note that the mapping χ_+ takes values in $\widetilde{N}_+$ and the mapping ψ_+ takes values in $\widetilde{H}$. Moreover, one has

$$\partial_{+i} \chi_+ = 0, \qquad \partial_{+i} \psi_+ = 0. \tag{3.20}$$

Similarly, from the equality

$$\mu_+' \nu_-' \eta' \gamma_-' = a \mu_+ \nu_- \eta \gamma_-$$

we get the relation

$$\mu_+' = a \mu_+ \chi_- \psi_-, \tag{3.21}$$

with the mappings χ_- and ψ_- given by

$$\chi_- = \nu_- \eta \gamma_- \gamma_-'^{-1} \eta'^{-1} \nu_-'^{-1} \eta' \gamma_-' \gamma_-^{-1} \eta^{-1}, \qquad \psi_- = \eta \gamma_- \gamma_-'^{-1} \eta'^{-1}.$$

Here the mapping χ_- take values in $\widetilde{N}_-$, the mapping ψ_- take values in $\widetilde{H}$, and one has

$$\partial_{-i} \chi_- = 0, \qquad \partial_{-i} \psi_- = 0. \tag{3.22}$$

Now using the Gauss decompositions

$$\mu_+^{-1} \mu_- = \nu_- \eta \nu_+^{-1}, \qquad \mu_+'^{-1} \mu_-' = \nu_-' \eta' \nu_+'^{-1}$$

and relations (3.19), (3.21), one comes to the equalities

$$\eta' = \psi_-^{-1} \eta \psi_+, \qquad \nu_\pm' = \psi_\pm^{-1} \chi_\pm^{-1} \nu_\pm \psi_\pm. \tag{3.23}$$

Further, from the definition of the mapping ψ_+ one gets

$$\gamma_-' = \psi_+^{-1} \gamma_-. \tag{3.24}$$

Since $\gamma' = \gamma$, we can write

$$\gamma_+'^{-1} \eta' \gamma_-' = \gamma_+^{-1} \eta \gamma_-,$$

therefore,

$$\gamma_+' = \psi_-^{-1} \gamma_+. \tag{3.25}$$

Equalities (3.19) and (3.21) give the relation

$$\begin{aligned} &\mu_\pm'^{-1} \partial_{\pm i} \mu_\pm' \\ &= \psi_\mp^{-1} \chi_\mp^{-1} (\mu_\pm^{-1} \partial_{\pm i} \mu_\pm) \chi_\mp \psi_\mp + \psi_\mp^{-1} (\chi_\mp^{-1} \partial_{\pm i} \chi_\mp) \psi_\mp + \psi_\mp^{-1} \partial_{\pm i} \psi_\mp, \end{aligned} \tag{3.26}$$

which implies

$$\sum_{m=\pm 1}^{\pm l_\pm \mp 1} \lambda_{\pm i, m}' = \left[\psi_\mp^{-1} \chi_\mp^{-1} \left(\sum_{m=\pm 1}^{\pm l_\pm \mp 1} \lambda_{\pm i, m} \right) \chi_\mp \psi_\mp \right]_{\widetilde{\mathfrak{n}}_\pm}. \tag{3.27}$$

Let again $\gamma_\pm$, $\lambda_{\pm i,m}$ and $\gamma_\pm'$, $\lambda_{\pm i,m}'$ be two sets of mappings satisfying the integrability conditions of the equations determining the corresponding mappings $\mu_\pm$ and $\mu_\pm'$. Denote

by γ, $\upsilon_{\pm i,m}$ and by γ', $\upsilon'_{\pm i,m}$ the corresponding solutions of the multidimensional Toda type equations. Suppose that the mappings $\mu_\pm$ and $\mu'_\pm$ are connected by relations (3.19) and (3.21) where the mappings $\chi_\pm$ take values in $\widetilde{N}_\pm$ and the mappings $\psi_\pm$ take values in $\widetilde{H}$. It is not difficult to get convinced that the mappings $\chi_\pm$ and $\psi_\pm$ satisfy relations (3.20) and (3.22). It is also clear that in the case under consideration relations (3.23) and (3.26) are valid. From (3.26) it follows that

$$\gamma'_\pm c_{\pm i} \gamma'^{-1}_\pm = \psi^{-1}_\mp \gamma_\pm c_{\pm i} \gamma^{-1}_\pm \psi_\mp.$$

Therefore, one has

$$\gamma'_\pm = \psi^{-1}_\mp \gamma_\pm \xi_\pm, \tag{3.28}$$

where the mappings $\xi_\pm$ take values in $\widetilde{H}_\pm$. Taking into account (3.23), we get

$$\gamma' = \xi^{-1}_+ \gamma \xi_-.$$

Using now (3.8), (3.9) and the similar relations for the mappings $\upsilon'_{\pm i,m}$, we come to the relations

$$\upsilon'_{\pm i,m} = \xi^{-1}_\pm \upsilon_{\pm i,m} \xi_\pm.$$

If instead of (3.28) one has (3.24) and (3.25), then $\gamma' = \gamma$ and $\upsilon'_{\pm i,m} = \upsilon_{\pm i,m}$. Thus, the sets $\gamma_\pm$, $\lambda_{\pm i,m}$ and $\gamma'_\pm$, $\lambda'_{\pm i,m}$ give the same solution of the multidimensional Toda type equations if and only if the corresponding mappings $\mu_\pm$ and $\mu'_\pm$ are connected by relations (3.19), (3.21) and equalities (3.24), (3.25) are valid.

Let now $\gamma_\pm$ and $\lambda_{\pm i,m}$ be a set of integration data, and $\mu_\pm$ be the solution of equations (3.1), (3.5) and (3.2), (3.6) specified by initial conditions (3.7). Suppose that the mappings $\mu_\pm$ admit the Gauss decompositions

$$\mu_\pm = \mu'_\pm \nu'_\mp \eta'_\mp. \tag{3.29}$$

where the mappings $\mu'_\pm$ take values in $a'_\pm \widetilde{N}_\pm$, the mappings $\nu'_\pm$ take values in $\widetilde{N}_\pm$ and the mappings $\eta'_\pm$ take values in $\widetilde{H}$. Note that if $a_\pm \widetilde{N}_\pm = a'_\pm \widetilde{N}_\pm$, then $\mu'_\pm = \mu_\pm$. Equalities (3.29) imply that the mappings $\mu_\pm$ and $\mu'_\pm$ are connected by relations (3.19) and (3.21) with $a = e$ and

$$\chi_\pm = \eta'^{-1}_\pm \nu'^{-1}_\pm \eta'_\pm, \qquad \psi_\pm = \eta'^{-1}_\pm.$$

From (3.26) it follows that the mappings $\gamma'_\pm$ and $\lambda'_{\pm i,m}$ given by (3.24), (3.25) and (3.27) generate the mappings $\mu'_\pm$ as a solution of equations (3.1), (3.2). It is clear that in the case under consideration the solutions of the multidimensional Toda type equations, obtained using the mappings $\gamma_\pm$, $\lambda_{\pm i,m}$ and $\gamma'_\pm$, $\lambda'_{\pm i,m}$, coincide. Certainly, we must use here the appropriate initial conditions for the mappings $\mu_\pm$ and $\mu'_\pm$. Thus, we see that the solution of the multidimensional Toda equation, which is determined by the mappings $\gamma_\pm$, $\lambda_{\pm i,m}$ and by the corresponding mappings $\mu_\pm$ taking values in $a_\pm \widetilde{N}_\pm$, can be also obtained starting from some mappings $\gamma'_\pm$, $\lambda'_{\pm i,m}$ and the corresponding mappings $\mu'_\pm$ taking values in $a'_\pm \widetilde{N}_\pm$. The above construction fails when the mappings $\mu_\pm$ do not admit Gauss decomposition (3.29).

Roughly speaking, almost all solutions of the multidimensional Toda type equations can be obtained by the method described in the present section if we will use only the mappings

$\mu_\pm$ taking values in the sets $a_\pm \widetilde{N}_\pm$ for some fixed elements $a_\pm \in G$. In particular, we can consider only the mappings $\mu_\pm$ taking values in $\widetilde{N}_\pm$.

Summarising our consideration, describe once more the procedure for obtaining the general solution to the multidimensional Toda type equations. We start with the mappings $\gamma_\pm$ and $\lambda_{\pm i,m}$ which satisfy (3.4) and the integrability conditions of equations (3.1), (3.5) and (3.2), (3.6). Integrating these equations, we get the mappings $\mu_\pm$. Further, Gauss decomposition (3.3) gives the mappings η and $\nu_\pm$. Finally, using (2.29), (3.8) and (3.9), we obtain the mappings γ and $\upsilon_{\pm i,m}$ which satisfy the multidimensional Toda type equations. Any solution can be obtained by using this procedure. Two sets of mappings $\gamma_\pm$, $\lambda_{\pm i,m}$ and $\gamma'_\pm$, $\lambda'_{\pm i,m}$ give the same solution if and only if the corresponding mappings $\mu_\pm$ and $\mu'_\pm$ are connected by relations (3.19), (3.21) and equalities (3.24), (3.25) are valid. Almost all solutions of the multidimensional Toda type equations can be obtained using the mappings $\mu_\pm$ taking values in the subgroups $\widetilde{N}_\pm$.

3.4 Automorphisms and reduction

Let Σ be an automorphism of the Lie group G, and σ be the corresponding automorphism of the Lie algebra $\mathfrak{g}$. Suppose that

$$\sigma(\mathfrak{g}_m) = \mathfrak{g}_m. \tag{3.30}$$

In this case

$$\Sigma(\widetilde{H}) = \widetilde{H}, \qquad \Sigma(\widetilde{N}_\pm) = \widetilde{N}_\pm. \tag{3.31}$$

Suppose additionally that

$$\sigma(c_{\pm i}) = c_{\pm i}. \tag{3.32}$$

It is easy to show now that if mappings γ and $\upsilon_{\pm i,m}$ satisfy the multidimensional Toda type equations, then the mappings $\Sigma \circ \gamma$ and $\sigma \circ \upsilon_{\pm i,m}$ satisfy the same equations. In such a situation we can consider the subset of the solutions satisfying the conditions

$$\Sigma \circ \gamma = \gamma, \qquad \sigma \circ \upsilon_{\pm i,m} = \upsilon_{\pm i,m}. \tag{3.33}$$

It is customary to call the transition to some subset of the solutions of a given system of equations a reduction of the system. Below we discuss a method to obtain solutions of the multidimensional Toda type system satisfying relations (3.33). Introduce first some notations and give a few definitions.

Denote by $\widehat{G}$ the subgroup of G formed by the elements invariant with respect to the automorphism Σ. In other words,

$$\widehat{G} = \{a \in G \mid \Sigma(a) = a\}.$$

The subgroup $\widehat{G}$ is a closed subgroup of G. Therefore, $\widehat{G}$ is a Lie subgroup of G. It is clear that the subalgebra $\widehat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$, defined by

$$\widehat{\mathfrak{g}} = \{x \in \mathfrak{g} \mid \sigma(x) = x\},$$

is the Lie algebra of $\widehat{G}$. The Lie algebra $\widehat{\mathfrak{g}}$ is a $\mathbb{Z}$–graded subalgebra of $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \widehat{\mathfrak{g}}_m,$$

where

$$\widehat{\mathfrak{g}}_m = \{x \in \mathfrak{g}_m \mid \sigma(x) = x\}.$$

Define now the following Lie subgroups of $\widehat{G}$,

$$\widehat{\widetilde{H}} = \{a \in \widetilde{H} \mid \Sigma(a) = a\}, \qquad \widehat{\widetilde{N}}_\pm = \{a \in \widetilde{N}_\pm \mid \Sigma(a) = a\}.$$

Using the definitions given above, we can reformulate conditions (3.33) by saying that the mapping γ takes value in $\widehat{\widetilde{H}}$, and the mappings $\upsilon_{\pm i,m}$ take values in $\widehat{\mathfrak{g}}_m$.

Let a be an arbitrary element of $\widehat{G}$. Consider a as an element of G and suppose that it has the Gauss decomposition (2.9). Then from the equality $\Sigma(a) = a$, we get the relation

$$\Sigma(n_-)\Sigma(h)\Sigma(n_+^{-1}) = n_- h n_+^{-1}.$$

Taking into account (3.31) and the uniqueness of the Gauss decomposition (2.9), we conclude that

$$\Sigma(h) = h, \qquad \Sigma(n_\pm) = n_\pm.$$

Thus, the elements of some dense subset of $\widehat{G}$ possess the Gauss decomposition (2.9) with $h \in \widehat{\widetilde{H}}$, $n_\pm \in \widehat{\widetilde{N}}_\pm$, and this decomposition is unique. Similarly, one can get convinced that any element of $\widehat{G}$ has the modified Gauss decompositions (2.10) with $m_\pm \in a_\pm \widehat{\widetilde{N}}_\pm$ for some elements $a_\pm \in \widehat{G}$, $n_\pm \in \widehat{\widetilde{N}}_\pm$ and $h_\pm \in \widehat{\widetilde{H}}$.

To obtain solutions of the multidimensional Toda type equations satisfying (3.33), we start with the mappings $\gamma_\pm$ and $\lambda_{\pm i,m}$ which satisfy the corresponding integrability conditions and the relations similar to (3.33):

$$\Sigma \circ \gamma_\pm = \gamma_\pm, \qquad \sigma \circ \lambda_{\pm i,m} = \lambda_{\pm i,m}. \tag{3.34}$$

In this case, for any solution of equations (3.1), (3.5) and (3.2), (3.6) one has

$$\sigma \circ (\mu_\pm^{-1} \partial_{\pm i} \mu_\pm) = \mu_\pm^{-1} \partial_{\pm i} \mu_\pm, \qquad \sigma \circ (\mu_\pm^{-1} \partial_{\mp i} \mu_\pm) = \mu_\pm^{-1} \partial_{\mp i} \mu_\pm.$$

From these relations it follows that

$$\Sigma \circ \mu_\pm = b_\pm \mu_\pm, \tag{3.35}$$

where $b_\pm$ are some elements of G. Recall that a solution of equations (3.1), (3.5) and (3.2), (3.6) is uniquely specified by conditions (3.7). If the elements $a_\pm$ entering these conditions belong to the group $\widehat{G}$, then instead of (3.35), we get for the corresponding mappings $\mu_\pm$ the relations

$$\Sigma \circ \mu_\pm = \mu_\pm.$$

For such mappings $\mu_\pm$ the Gauss decomposition (3.3) gives the mappings η and $\nu_\pm$ which satisfy the equalities

$$\Sigma \circ \eta = \eta, \qquad \Sigma \circ \nu_\pm = \nu_\pm.$$

It is not difficult to get convinced that the corresponding solution of the multidimensional Toda type equations satisfies (3.33). Show now that any solution of the multidimensional Toda type equations satisfying (3.33) can be obtained in such a way.

Let mappings γ and $\upsilon_{\pm i,m}$ satisfy the multidimensional Toda type equations and equalities (3.33) are valid. In this case, for the flat connection ω with the components defined by (2.22) and (2.23), one has

$$\sigma \circ \omega = \omega.$$

Therefore, a mapping $\varphi : M \to G$ generating the connection ω satisfies, in general, the relation

$$\Sigma \circ \varphi = b\varphi,$$

where b is some element of G. However, if for some point $p \in M$, one has $\varphi(p) \in \widehat{G}$, then we have the relation

$$\Sigma \circ \varphi = \varphi. \tag{3.36}$$

Since the mapping φ is defined up to the multiplication from the left hand side by an arbitrary element of G, it is clear that we can always choose this mapping in such a way that it satisfies (3.36). Take such a mapping φ and construct for it the local Gauss decompositions (2.11) where the mappings $\mu_\pm$ take values in the sets $a_\pm \widehat{\widetilde{N}}_\pm$ for some $a_\pm \in \widehat{G}$, the mappings $\nu_\pm$ take values in $\widehat{\widetilde{N}}_\pm$, and the mappings $\eta_\pm$ take values in $\widehat{\widetilde{H}}$. In particular, one has

$$\Sigma \circ \mu_\pm = \mu_\pm. \tag{3.37}$$

As it follows from the consideration performed in section 3.2, the mappings $\mu_\pm$ can be treated as solutions of equations (3.1), (3.5) and (3.2), (3.6) for some mappings $\lambda_{\pm i,m}$ and the mappings $\gamma_\pm$ given by (3.16), (3.17). Clearly, in this case

$$\Sigma \circ \gamma_\pm = \gamma_\pm,$$

and from (3.37) it follows that

$$\Sigma \circ \lambda_{\pm i,m} = \lambda_{\pm i,m}.$$

Moreover, the mappings $\gamma_\pm$ and $\lambda_{\pm i,m}$ are integration data leading to the considered solution of the multidimensional Toda type equations. Thus, if we start with mappings $\gamma_\pm$ and $\lambda_{\pm i,m}$ which satisfy the integrability conditions and relations (3.34), use the mappings $\mu_\pm$ specified by conditions (3.7) with $a_\pm \in \widehat{G}$, we get a solution satisfying (3.33), and any such a solution can be obtained in this way.

Let now Σ be an antiautomorphism of G, and σ be the corresponding antiautomorphism of $\mathfrak{g}$. In this case we again suppose the validity of the relations $\sigma(\mathfrak{g}_m) = \mathfrak{g}_m$ which imply that $\Sigma(\widetilde{H}) = \widetilde{H}$ and $\Sigma(\widetilde{N}_\pm) = \widetilde{N}_\pm$. However, instead of (3.32), we suppose that

$$\sigma(c_{\pm i}) = -c_{\pm i}.$$

One can easily get convinced that if the mappings γ and $\upsilon_{\pm i,m}$ satisfy the multidimensional Toda type equations, then the mappings $(\Sigma \circ \gamma)^{-1}$ and $-\sigma \circ \upsilon_{\pm i,m}$ also satisfy these equations. Therefore, it is natural to consider the reduction to the mappings satisfying the conditions

$$\Sigma \circ \gamma = \gamma^{-1}, \qquad \sigma \circ \upsilon_{\pm i,m} = -\upsilon_{\pm i,m}.$$

The subgroup $\widehat{G}$ is defined now as

$$\widehat{G} = \{a \in G \mid \Sigma(a) = a^{-1}\}. \tag{3.38}$$

To get the general solution of the reduced system, we should start with the integration data $\gamma_\pm$ and $\lambda_{\pm i,m}$ which satisfy the relations

$$\Sigma \circ \gamma_\pm = \gamma_\pm^{-1}, \qquad \sigma \circ \lambda_{\pm i,m} = -\lambda_{\pm i,m},$$

and use the mappings $\mu_\pm$ specified by conditions (3.7) with $a_\pm$ belonging to the subgroup $\widehat{G}$ defined by (3.38).

One can also consider reductions based on antiholomorphic automorphisms of G and on the corresponding antilinear automorphisms of $\mathfrak{g}$. In this way it is possible to introduce the notion of 'real' solutions to multidimensional Toda type system. We refer the reader to the discussion of this problem given in [16, 17] for the two dimensional case. The generalisation to the multidimensional case is straightforward.

4 Examples

4.1 Generalised WZNW equations

The simplest example of the multidimensional Toda type equations is the so called generalised Wess–Zumino–Novikov–Witten (WZNW) equations [11]. Let G be an arbitrary complex connected matrix Lie group. Consider the Lie algebra $\mathfrak{g}$ of G as a $\mathbb{Z}$–graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$, where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{\pm 1} = \{0\}$. In this case the subgroup $\widetilde{H}$ coincides with the whole Lie group G, and the subgroups $\widetilde{N}_\pm$ are trivial. So, the mapping γ parametrising the connection components of form (2.22), (2.23), takes values in G. The only possible choice for the elements $c_{\pm i}$ is $c_{\pm i} = 0$, and equations (2.35)–(2.37) take the form

$$\partial_{+j}(\gamma^{-1}\partial_{-i}\gamma) = 0,$$

which can be also rewritten as

$$\partial_{-i}(\partial_{+j}\gamma\gamma^{-1}) = 0.$$

These are the equations which are called in [11] the *generalised WZNW equations*. They are, in a sense, trivial and can be easily solved. However, in a direct analogy with two dimensional case, see, for example, [10], it is possible to consider the multidimensional Toda type equations as reductions of the generalised WZNW equations.

Let us show how our general integration scheme works in this simplest case. We start with the mappings $\gamma_\pm$ which take values in $\widetilde{H} = G$ and satisfy the relations

$$\partial_{\mp i}\gamma_\pm = 0.$$

For the mappings $\mu_\pm$ we easily find

$$\mu_\pm = a_\pm,$$

where $a_\pm$ are some arbitrary elements of G. The Gauss decomposition (3.3) gives $\eta = a_+^{-1}a_-$, and for the general solution of the generalised WZNW equations we have

$$\gamma = \gamma_+^{-1}a_+^{-1}a_-\gamma_-.$$

It is clear that the freedom to choose different elements $a_\pm$ is redundant, and one can put $a_\pm = e$, which gives the usual expression for the general solution

$$\gamma = \gamma_+^{-1}\gamma_-.$$

4.2 Example based on Lie group $\mathrm{GL}(m, \mathbb{C})$

Recall that the Lie group $\mathrm{GL}(m, \mathbb{C})$ consists of all nondegenerate $m \times m$ complex matrices. This group is reductive. We identify the Lie algebra of $\mathrm{GL}(m, \mathbb{C})$ with the Lie algebra $\mathfrak{gl}(m, \mathbb{C})$.

Introduce the following $\mathbb{Z}$–gradation of $\mathfrak{gl}(m, \mathbb{C})$. Let n and k be some positive integers such that $m = n + k$. Consider a general element x of $\mathfrak{gl}(m, \mathbb{C})$ as a 2×2 block matrix

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $n\times n$ matrix, B is an $n\times k$ matrix, C is a $k\times n$ matrix, and D is a $k\times k$ matrix. Define the subspace $\mathfrak{g}_0$ as the subspace of $\mathfrak{gl}(m, \mathbb{C})$, consisting of all block diagonal matrices, the subspaces $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$ as the subspaces formed by all strictly lower and upper triangular block matrices, respectively.

Consider the multidimensional Toda type equations (2.35)–(2.37) which correspond to the choice $l_{-i} = l_{+i} = 1$. In our case the general form of the elements $c_{\pm i}$ is

$$c_{-i} = \begin{pmatrix} 0 & 0 \\ C_{-i} & 0 \end{pmatrix}, \qquad c_{+i} = \begin{pmatrix} 0 & C_{+i} \\ 0 & 0 \end{pmatrix},$$

where C_{-i} are $k\times n$ matrices, and C_{+i} are $n\times k$ matrices. Since $\mathfrak{g}_{\pm 2} = \{0\}$, then conditions (2.18) are satisfied. The subgroup $\widetilde{H}$ is isomorphic to the group $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$, and the mapping γ has the block diagonal form

$$\gamma = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where the mappings β_1 and β_2 take values in $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(k, \mathbb{C})$, respectively. It is not difficult to show that

$$\gamma c_{-i} \gamma^{-1} = \begin{pmatrix} 0 & 0 \\ \beta_2 C_{-i} \beta_1^{-1} & 0 \end{pmatrix};$$

hence, equations (2.35) take the following form:

$$\partial_{-i}(\beta_2 C_{-j} \beta_1^{-1}) = \partial_{-j}(\beta_2 C_{-i} \beta_1^{-1}). \tag{4.1}$$

Similarly, using the relation

$$\gamma^{-1} c_{+i} \gamma = \begin{pmatrix} 0 & \beta_1^{-1} C_{+i} \beta_2 \\ 0 & 0 \end{pmatrix},$$

we represent equations (2.36) as

$$\partial_{+i}(\beta_1^{-1} C_{+j} \beta_2) = \partial_{+j}(\beta_1^{-1} C_{+i} \beta_2). \tag{4.2}$$

Finally, equations (2.37) take the form

$$\partial_{+j}(\beta_1^{-1} \partial_{-i} \beta_1) = -\beta_1^{-1} C_{+j} \beta_2 C_{-i}, \tag{4.3}$$
$$\partial_{+j}(\beta_2^{-1} \partial_{-i} \beta_2) = C_{-i} \beta_1^{-1} C_{+j} \beta_2. \tag{4.4}$$

In accordance with our integration scheme, to construct the general solution for equations (4.1)–(4.4) we should start with the mappings $\gamma_\pm$ which take values in $\widetilde{H}$ and satisfy (3.4). Write for these mappings the block matrix representation

$$\gamma_\pm = \begin{pmatrix} \beta_{\pm 1} & 0 \\ 0 & \beta_{\pm 2} \end{pmatrix}.$$

Recall that almost all solutions of the multidimensional Toda type equations can be obtained using the mappings $\mu_\pm$ taking values in the subgroups $\widetilde{N}_\pm$. Therefore, we choose these mappings in the form

$$\mu_- = \begin{pmatrix} I_n & 0 \\ \mu_{-21} & I_k \end{pmatrix}, \qquad \mu_+ = \begin{pmatrix} I_n & \mu_{+12} \\ 0 & I_k \end{pmatrix},$$

where μ_{-21} and μ_{+12} take values in the spaces of $k \times n$ and $n \times k$ matrices, respectively. Equations (3.1), (3.5) and (3.2), (3.6) are reduced now to the equations

$$\partial_{-i}\mu_{-21} = \beta_{-2} C_{-i} \beta_{-1}^{-1}, \qquad \partial_{+i}\mu_{-21} = 0, \tag{4.5}$$
$$\partial_{+i}\mu_{+12} = \beta_{+1} C_{+i} \beta_{+2}^{-1}, \qquad \partial_{-i}\mu_{+12} = 0. \tag{4.6}$$

The corresponding integrability conditions are

$$\partial_{-i}(\beta_{-2} C_{-j} \beta_{-1}^{-1}) = \partial_{-j}(\beta_{-2} C_{-i} \beta_{-1}^{-1}), \tag{4.7}$$
$$\partial_{+i}(\beta_{+1} C_{+j} \beta_{+2}^{-1}) = \partial_{+j}(\beta_{+1} C_{+i} \beta_{+2}^{-1}). \tag{4.8}$$

Here we will not study the problem of solving the integrability conditions for a general choice of n, k and $C_{\pm i}$. In the end of this section we discuss a case when it is quite easy to find explicitly all the mappings $\gamma_\pm$ satisfying the integrability conditions, while now we will continue the consideration of the integration procedure for the general case.

Suppose that the mappings $\gamma_\pm$ satisfy the integrability conditions and we have found the corresponding mappings $\mu_\pm$. Determine from the Gauss decomposition (3.3) the mappings $\nu_\pm$ and η. Actually, in the case under consideration we need only the mapping η. Using for the mappings ν_-, ν_+ and η the following representations

$$\nu_- = \begin{pmatrix} I_n & 0 \\ \nu_{-21} & I_k \end{pmatrix}, \qquad \nu_+ = \begin{pmatrix} I_n & \nu_{+12} \\ 0 & I_k \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_{11} & 0 \\ 0 & \eta_{22} \end{pmatrix},$$

we find that

$$\nu_{-21} = \mu_{-21}(I_n - \mu_{+12}\mu_{-21})^{-1}, \qquad \nu_{+12} = (I_n - \mu_{+12}\mu_{-21})^{-1}\mu_{+12},$$
$$\eta_{11} = I_n - \mu_{+12}\mu_{-21}, \qquad \eta_{22} = I_k + \mu_{-21}(I_n - \mu_{+12}\mu_{-21})^{-1}\mu_{+12}.$$

It is worth to note here that the mapping $\mu_+^{-1}\mu_-$ has the Gauss decomposition (3.3) only at those points p of M, for which

$$\det\,(I_n - \mu_{+12}(p)\mu_{-21}(p)) \neq 0. \tag{4.9}$$

Now, using relation (2.29), we get for the general solution of system (4.1)–(4.4) the following expression:

$$\beta_1 = \beta_{+1}^{-1}(I_n - \mu_{+12}\mu_{-21})\beta_{-1},$$
$$\beta_2 = \beta_{+2}^{-1}(I_k + \mu_{-21}(I_n - \mu_{+12}\mu_{-21})^{-1}\mu_{+12})\beta_{-2}.$$

Consider now the case when $n = m - 1$. In this case β_1 takes values in $\mathrm{GL}(n, \mathbb{C})$, β_2 is a complex function, C_{-i} and C_{+i} are $1\times n$ and $n\times 1$ matrices, respectively. Suppose that the dimension of the manifold M is equal to $2n$ and define $C_{\pm i}$ by

$$(C_{\pm i})_r = \delta_{ir}.$$

System (4.1)–(4.4) takes now the form

$$\partial_{-i}(\beta_2(\beta_1^{-1})_{jr}) = \partial_{-j}(\beta_2(\beta_1^{-1})_{ir}), \tag{4.10}$$
$$\partial_{+i}((\beta_1^{-1})_{rj}\beta_2) = \partial_{+j}((\beta_1^{-1})_{ri}\beta_2), \tag{4.11}$$
$$\partial_{+j}(\beta_1^{-1}\partial_{-i}\beta_1)_{rs} = -(\beta_1^{-1})_{rj}\beta_2\delta_{is}, \tag{4.12}$$
$$\partial_{+j}(\beta_2^{-1}\partial_{-i}\beta_2) = (\beta_1^{-1})_{ij}\beta_2, \tag{4.13}$$

and the integrability conditions (4.7), (4.8) can be rewritten as

$$\partial_{-i}(\beta_{-2}(\beta_{-1}^{-1})_{jr}) = \partial_{-j}(\beta_{-2}(\beta_{-1}^{-1})_{ir}),$$
$$\partial_{+i}((\beta_{+1})_{rj}\beta_{+2}^{-1}) = \partial_{+j}((\beta_{+1})_{ri}\beta_{+2}^{-1}).$$

The general solution for these integrability conditions is

$$(\beta_{-1}^{-1})_{ir} = U_-\partial_{-i}V_{-r}, \qquad \beta_{-2}^{-1} = U_-,$$
$$(\beta_{+1})_{ri} = U_+\partial_{+i}V_{+r}, \qquad \beta_{+2} = U_+.$$

Here $U_\pm$ and $V_{\pm r}$ are arbitrary functions satisfying the conditions

$$\partial_\mp U_\pm = 0, \qquad \partial_\mp V_{\pm r} = 0.$$

Moreover, for any point p of M one should have

$$U_\pm(p) \neq 0, \qquad \det(\partial_{\pm i}V_{\pm r}(p)) \neq 0.$$

The general solution of equations (4.5), (4.6) is

$$\mu_{-21} = V_-, \qquad \mu_{+12} = V_+,$$

where V_- is the $1\times n$ matrix valued function constructed with the functions V_{-r}, and V_+ is the $n\times 1$ matrix valued function constructed with the functions V_{+r}. Thus, we have

$$\eta_{11} = I_n - V_+V_-.$$

In the case under consideration, condition (4.9) which guarantees the existence of the Gauss decomposition (3.3), is equivalent to

$$1 - V_-(p)V_+(p) \neq 0.$$

When this condition is satisfied, one has

$$(I_n - \mu_{+12}\mu_{-21})^{-1} = (I_n - V_+V_-)^{-1} = I_n + \frac{1}{1 - V_-V_+}V_+V_-,$$

and, therefore,

$$\eta_{22} = \frac{1}{1 - V_-V_+}.$$

Taking the above remarks into account, we come to the following expressions for the general solution of system (4.10)–(4.13):

$$(\beta_1^{-1})_{ij} = -U_+U_-(1 - V_-V_+)\partial_{-i}\partial_{+j}\ln(1 - V_-V_+),$$
$$\beta_2^{-1} = U_+U_-(1 - V_-V_+).$$

4.3 Cecotti–Vafa type equations

In this example we discuss the multidimensional Toda system associated with the loop group $\mathcal{L}(\mathrm{GL}(m,\mathbb{C}))$ which is an infinite dimensional Lie group defined as the group of smooth mappings from the circle S^1 to the Lie group $\mathrm{GL}(m,\mathbb{C})$. We think of the circle as consisting of complex numbers ζ of modulus one. The Lie algebra of $\mathcal{L}(\mathrm{GL}(m,\mathbb{C}))$ is the Lie algebra $\mathcal{L}(\mathfrak{gl}(m,\mathbb{C}))$ consisting of smooth mappings from S^1 to the Lie algebra $\mathfrak{gl}(m,\mathbb{C})$.

In the previous section we considered some class of $\mathbb{Z}$–gradations of the Lie algebra $\mathfrak{gl}(m,\mathbb{C})$ based on the representation of m as the sum of two positive integers n and k. Any such a gradation can be extended to a $\mathbb{Z}$–gradation of the loop algebra $\mathcal{L}(\mathrm{GL}(m,\mathbb{C})$. Here we restrict ourselves to the case $m = 2n$. In this case the element

$$q = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

of $\mathfrak{gl}(2n,\mathbb{C})$ is the grading operator of the $\mathbb{Z}$–gradation under consideration. This means that an element x of $\mathfrak{gl}(2n,\mathbb{C})$ belongs to the subspace $\mathfrak{g}_k$ if and only if $[q,x] = kx$. Using the operator q, we introduce the following $\mathbb{Z}$–gradation of $\mathcal{L}(\mathfrak{gl}(2n,\mathbb{C}))$. The subspace $\mathfrak{g}_k$ of $\mathcal{L}(\mathfrak{gl}(2n,\mathbb{C}))$ is defined as the subspace formed by the elements $x(\zeta)$ of $\mathcal{L}(\mathfrak{gl}(2n,\mathbb{C}))$ satisfying the relation

$$[q, x(\zeta)] + 2\zeta\frac{dx(\zeta)}{d\zeta} = kx(\zeta).$$

In particular, the subspaces $\mathfrak{g}_0$, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$ of $\mathcal{L}(\mathfrak{gl}(2n,\mathbb{C}))$ consist respectively of the elements

$$x(\zeta) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad x(\zeta) = \begin{pmatrix} 0 & \zeta^{-1}B \\ C & 0 \end{pmatrix}, \qquad x(\zeta) = \begin{pmatrix} 0 & B \\ \zeta C & 0 \end{pmatrix},$$

where A, B, C and D are arbitrary $n\times n$ matrices which do not depend on ζ.

Consider the multidimensional Toda type equations (2.35)–(2.37) which correspond to the choice $l_{-i} = l_{+i} = 1$. In this case the general form of the elements $c_{\pm i}$ is

$$c_{-i} = \begin{pmatrix} 0 & \zeta^{-1}B_{-i} \\ C_{-i} & 0 \end{pmatrix}, \qquad c_{+i} = \begin{pmatrix} 0 & C_{+i} \\ \zeta B_{+i} & 0 \end{pmatrix}.$$

To satisfy conditions (2.18) we should have

$$B_{\pm i}C_{\pm j} - B_{\pm j}C_{\pm i} = 0, \quad C_{\pm i}B_{\pm j} - C_{\pm j}B_{\pm i} = 0.$$

The subgroup $\widetilde{H}$ is isomorphic to the group $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$, and the mapping γ has the block diagonal form

$$\gamma = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where β_1 and β_2 take values in $\mathrm{GL}(n,\mathbb{C})$. Hence, one obtains

$$\gamma c_{-i}\gamma^{-1} = \begin{pmatrix} 0 & \zeta^{-1}\beta_1 B_{-i}\beta_2^{-1} \\ \beta_2 C_{-j}\beta_1^{-1} & 0 \end{pmatrix},$$

and comes to following explicit expressions for equations (2.35):

$$\partial_{-i}(\beta_1 B_{-j}\beta_2^{-1}) = \partial_{-j}(\beta_1 B_{-i}\beta_2^{-1}), \tag{4.14}$$
$$\partial_{-i}(\beta_2 C_{-j}\beta_1^{-1}) = \partial_{-j}(\beta_2 C_{-i}\beta_1^{-1}). \tag{4.15}$$

Similarly, using the relation

$$\gamma^{-1}c_{+i}\gamma = \begin{pmatrix} 0 & \beta_1^{-1}C_{+i}\beta_2 \\ \zeta\beta_2^{-1}B_{+i}\beta_1 & 0 \end{pmatrix},$$

we can represent equations (2.36) as

$$\partial_{+i}(\beta_1^{-1}C_{+j}\beta_2) = \partial_{+j}(\beta_1^{-1}C_{+i}\beta_2), \tag{4.16}$$
$$\partial_{+i}(\beta_2^{-1}B_{+j}\beta_1) = \partial_{+j}(\beta_2^{-1}B_{+i}\beta_1). \tag{4.17}$$

Finally, equations (2.37) take the form

$$\partial_{+j}(\beta_1^{-1}\partial_{-i}\beta_1) = B_{-i}\beta_2^{-1}B_{+j}\beta_1 - \beta_1^{-1}C_{+j}\beta_2 C_{-i}, \tag{4.18}$$
$$\partial_{+j}(\beta_2^{-1}\partial_{-i}\beta_2) = C_{-i}\beta_1^{-1}C_{+j}\beta_2 - \beta_2^{-1}B_{+j}\beta_1 B_{-i}. \tag{4.19}$$

System (4.14)–(4.19) admits two interesting reductions, which can be defined with the help of the general scheme described in section 3.4. Represent an arbitrary element $a(\zeta)$ of $\mathcal{L}(\mathrm{GL}(2n,\mathbb{C}))$ in the block form,

$$a(\zeta) = \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix},$$

and define an automorphism Σ of $\mathcal{L}(\mathrm{GL}(2n,\mathbb{C}))$ by

$$\Sigma(a(\zeta)) = \begin{pmatrix} D(\zeta) & \zeta^{-1}C(\zeta) \\ \zeta B(\zeta) & A(\zeta) \end{pmatrix}.$$

It is clear that the corresponding automorphism σ of $\mathcal{L}(\mathfrak{gl}(2n,\mathbb{C}))$ is defined by the relation of the same form. In the case under consideration relation (3.30) is valid. Suppose that $B_{\pm i} = C_{\pm i}$, then relation (3.32) is also valid. Therefore, we can consider the reduction of

system (4.14)–(4.19) to the case when the mapping γ satisfies the equality $\Sigma \circ \gamma = \gamma$ which can be written as $\beta_1 = \beta_2$. The reduced system looks as

$$\partial_{-i}(\beta C_{-j}\beta^{-1}) = \partial_{-j}(\beta C_{-i}\beta^{-1}), \tag{4.20}$$

$$\partial_{+i}(\beta^{-1}C_{+j}\beta) = \partial_{+j}(\beta^{-1}C_{+i}\beta), \tag{4.21}$$

$$\partial_{+j}(\beta^{-1}\partial_{-i}\beta) = [C_{-i}, \beta^{-1}C_{+j}\beta], \tag{4.22}$$

where we have denoted $\beta = \beta_1 = \beta_2$.

The next reduction is connected with an antiautomorphism Σ of the group $\mathcal{L}(\mathrm{GL}(2n, \mathbb{C}))$ given by

$$\Sigma(a(\zeta)) = \begin{pmatrix} A(\zeta)^t & -\zeta^{-1}C(\zeta)^t \\ -\zeta B(\zeta)^t & D(\zeta)^t \end{pmatrix}.$$

The corresponding antiautomorphism of $\mathcal{L}(\mathfrak{gl}(2n, \mathbb{C}))$ is defined by the same formula. It is evident that $\sigma(\mathfrak{g}_k) = \mathfrak{g}_k$. Suppose that $B_{\pm i} = C^t_{\pm i}$, then $\sigma(c_{\pm i}) = -c_{\pm i}$, and one can consider the reduction of system (4.14)–(4.19) to the case when the mapping γ satisfies the equality $\Sigma \circ \gamma = \gamma^{-1}$ which is equivalent to the equalities $\beta_1^t = \beta_1^{-1}$, $\beta_2^t = \beta_2^{-1}$. The reduced system of equations can be written as

$$\partial_{-i}(\beta_2 C_{-j}\beta_1^t) = \partial_{-j}(\beta_2 C_{-i}\beta_1^t), \tag{4.23}$$

$$\partial_{+i}(\beta_1^t C_{+j}\beta_2) = \partial_{+j}(\beta_1^t C_{+i}\beta_2), \tag{4.24}$$

$$\partial_{+j}(\beta_1^t\partial_{-i}\beta_1) = C^t_{-i}\beta_2^t C^t_{+j}\beta_1 - \beta_1^t C_{+j}\beta_2 C_{-i}, \tag{4.25}$$

$$\partial_{+j}(\beta_2^t\partial_{-i}\beta_2) = C_{-i}\beta_1^t C_{+j}\beta_2 - \beta_2^t C^t_{+j}\beta_1 C^t_{-i}. \tag{4.26}$$

If simultaneously $B_{\pm i} = C_{\pm i}$ and $B_{\pm i} = C^t_{\pm i}$, one can perform both reductions. Here the reduced system has form (4.20)–(4.22) where the mapping β take values in the complex orthogonal group $\mathrm{O}(n, \mathbb{C})$. These are exactly the equations considered by S. Cecotti and C. Vafa [5]. As it was shown by B. A. Dubrovin [8] for $C_{-i} = C_{+i} = C_i$ with

$$(C_i)_{jk} = \delta_{ij}\delta_{jk},$$

the Cecotti–Vafa equations are connected with some well known equations in differential geometry. Actually, in [8] the case $M = \mathbb{C}^n$ was considered and an additional restriction $\beta^\dagger = \beta$ was imposed. Here equation (4.21) can be obtained from equation (4.20) by hermitian conjugation, and the system under consideration consists of equations (4.20) and (4.22) only. Rewrite equation (4.20) in the form

$$[\beta^{-1}\partial_{-i}\beta, C_j] = [\beta^{-1}\partial_{-j}\beta, C_i].$$

From this equation it follows that for some matrix valued mapping $b = (b_{ij})$, such that $b_{ij} = b_{ji}$, the relation

$$\beta^{-1}\partial_{-i}\beta = [C_i, b] \tag{4.27}$$

is valid. In fact, the right hand side of relation (4.27) does not contain the diagonal matrix elements of b, while the other matrix elements of B are uniquely determined by the left hand side of (4.27). Furthermore, relation (4.27) implies that the mapping b satisfies the equation

$$\partial_{-i}[C_j, b] - \partial_{-j}[C_i, b] + [[C_i, b], [C_j, b]] = 0. \tag{4.28}$$

From the other hand, if some mapping b satisfies equation (4.28), then there is a mapping β connected with b by relation (4.27), and such a mapping β satisfies equation (4.20). Therefore, system (4.20), (4.22) is equivalent to the system which consist of equations (4.27), (4.28) and the equation

$$\partial_{+j}[C_i, b] = [C_i, \beta^{-1} C_j \beta] \tag{4.29}$$

which follows from (4.27) and (4.22). Using the concrete form of the matrices C_i, one can write the system (4.27), (4.28) and (4.29) as

$$\partial_{-k} b_{ji} = b_{jk} b_{ki}, \qquad i,\ j,\ k \text{ distinct}; \tag{4.30}$$

$$\sum_{k=1}^{n} \partial_{-k} b_{ij} = 0; \qquad i \neq j; \tag{4.31}$$

$$\partial_{-i} \beta_{jk} = b_{ik} \beta_{ji}, \qquad i \neq j; \tag{4.32}$$

$$\sum_{k=1}^{n} \partial_{-k} \beta_{ij} = 0; \tag{4.33}$$

$$\partial_{+k} b_{ij} = \beta_{ki} \beta_{kj}, \qquad i \neq j. \tag{4.34}$$

Equations (4.30), (4.31) have the form of equations which provide vanishing of the curvature of the diagonal metric with symmetric rotation coefficients b_{ij} [7, 3]. Recall that such a metric is called a Egoroff metric. Note that the transition from system (4.20), (4.22) to system (4.30)–(4.34) is not very useful for obtaining solutions of (4.20), (4.22). A more constructive way here is to use the integration scheme described in section 3. Let us discuss the corresponding procedure for a more general system (4.23)–(4.26) with $C_{-i} = C_{+i} = C_i$.

The integrations data for system (4.23)–(4.26) consist of the mappings $\gamma_\pm$ having the following block diagonal form

$$\gamma_\pm = \begin{pmatrix} \beta_{\pm 1} & 0 \\ 0 & \beta_{\pm 2} \end{pmatrix}.$$

As it follows from the discussion given in section 3.4, the mappings $\beta_{\pm 1}$ and $\beta_{\pm 2}$ must satisfy the conditions

$$\beta_{\pm 1}^t = \beta_{\pm 1}^{-1}, \qquad \beta_{\pm 2}^t = \beta_{\pm 2}^{-1}.$$

The corresponding integrability conditions have the form

$$\partial_{\pm i}(\beta_{\pm 2} C_j \beta_{\pm 1}^t) = \partial_{\pm j}(\beta_{\pm 2} C_i \beta_{\pm 1}^t). \tag{4.35}$$

Rewriting these conditions as

$$\beta_{\pm 2}^t \partial_{\pm i} \beta_{\pm 2} C_j - C_j \beta_{\pm 1}^t \partial_{\pm i} \beta_{\pm 1} = \beta_{\pm 2}^t \partial_{\pm j} \beta_{\pm 2} C_i - C_i \beta_{\pm 1}^t \partial_{\pm j} \beta_{\pm 1},$$

we can get convinced that for some matrix valued mappings $b_\pm$ one has

$$\beta_{\pm 1}^t \partial_{\pm i} \beta_{\pm 1} = C_i b_\pm - b_\pm^t C_i, \qquad \beta_{\pm 2}^t \partial_{\pm i} \beta_{\pm 2} = C_i b_\pm^t - b_\pm C_i. \tag{4.36}$$

From these relations it follows that the mappings $b_\pm$ satisfy the equations

$$\partial_{\pm i}(b_\pm)_{ji} + \partial_{\pm j}(b_\pm)_{ij} + \sum_{k \neq i,j} (b_\pm)_{ik}(b_\pm)_{jk} = 0, \quad i \neq j; \tag{4.37}$$

$$\partial_{\pm k}(b_\pm)_{ji} = (b_\pm)_{jk}(b_\pm)_{ki}, \qquad i,\ j,\ k \text{ distinct}; \tag{4.38}$$

$$\partial_{\pm i}(b_\pm)_{ij} + \partial_{\pm j}(b_\pm)_{ji} + \sum_{k \neq i,j} (b_\pm)_{ki}(b_\pm)_{kj} = 0, \qquad i \neq j. \tag{4.39}$$

Conversely, if we have some mappings $b_\pm$ which satisfy equations (4.37)–(4.39), then there exist mappings $\beta_{\pm 1}$ and $\beta_{\pm 2}$ connected with $b_\pm$ by (4.36) and satisfying the integrability conditions (4.35).

System (4.37)–(4.39) represents a limiting case of the completely integrable Bourlet equations [7, 3] arising after an appropriate Inönü–Wigner contraction of the corresponding Lie algebra [18]. Sometimes this system is called the multidimensional generalised wave equations, while equation (4.22) is called the generalised sine–Gordon equation [2, 20, 1].

5 Outlook

Due to the algebraic and geometrical clearness of the equations discussed in the paper, we are firmly convinced that, in time, they will be quite relevant for a number of concrete applications in classical and quantum field theories, statistical mechanics and condensed matter physics. In support of this opinion we would like to remind a remarkable role of some special classes of the equations under consideration here.

Namely, in the framework of the standard abelian and nonabelian, conformal and affine Toda fields coupled to matter fields, some interesting physical phenomena which possibly can be described on the base of corresponding equations, are mentioned in [12, 9]. In particular, from the point of view of nonperturbative aspects of quantum field theories, they might be very useful for understanding the quantum theory of solitons, some confinement mechanisms for the quantum chromodynamics, electron–phonon systems, etc. Furthermore, the Cecotti–Vafa equations [5] of topological–antitopological fusion, which, as partial differential equations, are, in general, multidimensional ones, describe ground state metric of two dimensional $N = 2$ supersymmetric quantum field theory. As it was shown in [6], they are closely related to those for the correlators of the massive two dimensional Ising model, see, for example, [14] and references therein. This link is clarified in [6], in particular, in terms of the isomonodromic deformation in spirit of the holonomic field theory developed by the Japanese school, see, for example, [19].

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